Math 210C Lecture 26 Notes

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1 Tensor Product Representations, Character Tables, and Orthogonality Relations for Characters

1.1 Tensor product representations

Let G be a group, and let V, W be F-representations of G. We can form representations on

- $V \otimes_F W$: $g(v \otimes w) = gv \otimes gw$ and extended linearly.
- $\operatorname{Hom}_F(V, W)$: $(g \cdot \varphi)(v) = g\varphi(g^{-1}v)$.

Lemma 1.1. *Let G be a group, and let V, W be F-representations of G.*

1. $\chi_{V \otimes_F W} = \chi_V \chi_W$
2. $\chi_{\operatorname{Hom}_F(V,W)} = \overline{\chi_V} \cdot \chi_W$*, and* $\overline{\chi}_V(g) = \overline{\chi}_V(g^{-1})$.

Definition 1.1. If G is finite and $\operatorname{char}(G) \nmid |G|$, then we have **pairing** on characters of G:

$$\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi}(g) \chi'(g).$$

By the lemma,

$$\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\operatorname{Hom}_F(V,W)}(g).$$

Proposition 1.1. *Let V, W be finite dimensional F-representations of G. Then*

$$\langle \chi_V, \chi_W \rangle = \dim_F \operatorname{Hom}_{F[G]}(V, W).$$

Lemma 1.2. *Let V be a finite dimensional F-representation of G. Then*

$$\dim_F(V^G) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

Here is the proof of the lemma.

Proof. Let

$$e = \frac{1}{|G|} N_G \in F[G]$$

with $e^2 = e$. Then let $T : V \xrightarrow{e} V$. The minimal polynomial of T divides $x^2 - x = x(x-1)$. So T is diagonalizable, and the trace of T is the sum of the eigenvalues, which must all be 0 or 1. So $\mathrm{tr}(T)$ is the number of eigenvalues which are 1, namely $\dim(E_1(T))$.

Now observe that if $ev = v$, then $gv = gev = g\frac{1}{|G|}N_G v = ev = v$ for all $g \in G$. So $E_1(T) \subseteq V^G$. If $gv = v$ for all $g \in G$, then

$$ev = \frac{1}{|G|} N_G v = \frac{1}{|G|} |G| v = v,$$

so $v \in E_1(T)$. So $E_1(T) = V^G$. Then χ_V can be extended to $\chi_V : F[G] \to F$, and

$$\chi_V(e) = \mathrm{tr}(\rho_V(e)) = \mathrm{tr}(T) = \dim_F(E_1(T)).$$

By definition, we have

$$\chi_V(e) = \frac{1}{|G|} \sum_{g \in G} \chi_V(G).$$

Now set these equal. □

This now implies the proposition.

Proof. Observe that by the definition of the action of $G \circlearrowright \mathrm{Hom}_F(V, W)$, $\mathrm{Hom}_{F[G]}(V, W) = \mathrm{Hom}_F(V, W)^G$. □

1.2 Character tables

Let G be finite, let $g_1, \dots, f_r$ be representatives of conjugacy classes of F, and let $\chi_1, \dots, \chi_r$ be irreducible complex repreesntations of G. Let χ_i correspond to the representaiton V_i with $\dim_F(V_i) = n_i$.

Definition 1.2. The **character table** of G is a matrix in $M_r(\mathbb{C})$ with (i, j)-entry $\chi_i(g_j)$.

Example 1.1. Here is the character table for S_3:

S_3	e	$(1\ 2)$	$(1\ 2\ 3)$
χ_1	1	1	1
χ_{sgn}	1	-1	1
χ_3	2	0	-1

Example 1.2. Here is the character table for $\mathbb{Z}/n\mathbb{Z}$:

$\mathbb{Z}/n\mathbb{Z}$	1	1	2	$\cdots$	$n-1$
1	1	1	1	$\cdots$	1
χ	1	ζ	ζ^2	$\cdots$	ζ^{n-1}
χ^2	1	ζ^2	ζ^4	$\cdots$	$\zeta^{2(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ^{n-1}	1	ζ^{n-1}	$\zeta^{2(n-1)}$	$\cdots$	$\zeta^{(n-1)^2}$

1.3 Orthogonality relations for characters

Lemma 1.3. *Ket χ be a $\mathbb{C}$-valued character of G with degree d, $|G| = n$, and let $g \in G$. Then $\chi(g^{-1}) = \overline{\chi(g)}$, $\chi(g) \in \mathbb{Z}[\mu_n]$, and $|\chi(g)| \leq d$.*

Proof. Let χ correspond to $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(V) \cong \mathrm{GL}_d(\mathbb{C})$. Then $g^n = e$, and we can choose an isomorphism such that $\rho(g)$ is diagonal. So $\rho(g)^n = \mathrm{id}$, which means that the entries of $\rho(g)$ are in μ_n. If $\zeta \in \mu_n$, then $\overline{\zeta} = \zeta^{-1}$, so

$$\chi(g^{-1}) = \mathrm{tr}(\chi(g)^{-1}) = \overline{\mathrm{tr}(\rho(g))} = \overline{\chi(g)}.$$

Finally, we have that $\chi(g)$ is the sum of d n-th roots of unity. $\square$

Definition 1.3. An **inner product** on a $\mathbb{C}$-vector space V is an additive pairing $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that $\langle \alpha v, w \rangle = \alpha v, w$ and $\langle v, \beta w \rangle = \overline{\beta} \langle v, w \rangle$ for all $\alpha, \beta \in \mathbb{C}$.

Definition 1.4. An inner product is **positive definite** if $\langle v, v \rangle \geq 0$ for all $v \in V$ and $\langle v, v \rangle = 0 \iff v = 0$.

Definition 1.5. An inner product is **Hermitian** if $\langle v, w \rangle = \overline{\langle w, v \rangle}$

Definition 1.6. A basis B of V is **orthonormal** if $\langle v, w \rangle = \delta_{v,w}$ for all $v, w \in B$.

Definition 1.7. A finite dimensional $\mathbb{C}$-vector space with a Hermition pairing is called an **inner product space**.

If $c_1, \dots, v_n$ is a basis of a complex inner product space, let A be the matrix $A_{i,j} = \langle v_i, v_j \rangle$. Then $A = \overline{A}^\top$. It can be diagonalized by a **unitary** matrix (a matrix with $B\overline{B}^\top = I_n$).

Definition 1.8. An inner product on a $\mathbb{C}$-representation of G is G**-invariant** if $\langle gv, gw \rangle = \langle v, w \rangle$ for all $g \in G$.

Lemma 1.4. *There is a positive definite, Hermitian inner product on the space of $\mathbb{C}$-valued class functions on G given by*

$$\langle \theta, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \theta(g) \overline{\psi(g)}.$$

Theorem 1.1 (first orthogonality relation)**.** *The set of irreducible complex characters of G forms an orthonormal basis for the space of class functions.*

Proof. By Schur's lemma and our lemma from before,

$$\langle \chi_i, \chi_j \rangle = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}[G]}(V_i, V_j)) = \delta_{i,j}. \qquad \square$$

If we have rows $r_i, r_{i'}$ in the character table,

$$r_i \cdot r_{i'} = \frac{i}{|G|} \sum_{j=1}^{r} c_j \chi_i(g_j) \chi_{i'}(g_j),$$

where c_j is the order of the conjugacy class of g_j.

Theorem 1.2 (second orthogonality relation)**.**

$$\sum_{i=1}^{r} \chi_i(g) \overline{\chi_i(h)} = \begin{cases} |Z_g| & h \in C_g \\ 0 & h \notin C_g, \end{cases}$$

where Z_g is the centralizer, V_g is the conjugacy class of g, and $|Z_g| = n/|C_g|$.

Proof. Let A be the matrix with $A_{i,j} = \chi_i(g_j)$ (i.e. the character table). Let C be diagonal with (i,j)-entry $c_i = |C_{g_i}|$. THen

$$(AC\overline{A}^{\top})_{i,j} = \sum_{k=1}^{r} \chi_i(g_k) c_k \overline{\chi}_j(g_k) = \delta_{i,j}|G|.$$

The left hand side also equals

$$(AC\overline{A}^{\top})_{i,j} = (\overline{A}^{\top}AC)_{i,j} = \sum_{k=1}^{r} \overline{\chi_k(g_i)\chi_k(g_j)} c_j.$$

So we get

$$\sum_{k=1}^{r} \chi_k(g_i) \overline{\chi_k(g_j)} = \begin{cases} |Z_{g_i}| & g_i = g_j \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$